REGULARITY OF HOMOGENIZED BOUNDARY DATA IN
PERIODIC HOMOGENIZATION OF ELLIPTIC SYSTEMS

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DESCRIPTION OF THE SOLUTION

In [1], the authors considered the periodic homogenization of second-order elliptic systems
in divergence form with oscillating Dirichlet data or Neumann data of first order. They proved
that the homogenized boundary data belongs to $W^{1,p}$ for any $1 < p < \infty$. In particular,
this implies that the boundary layer tails are Hölder continuous of order $\alpha$ for any $\alpha \in (0, 1)$.

Precisely, we define the oscillating elliptic operator
\[
\mathcal{L}_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left\{a^{\alpha\beta}_{ij}(x/\varepsilon) \frac{\partial}{\partial x_j}\right\},
\]
We consider the Dirichlet problem
\[
\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \Omega, \quad \text{and} \quad u_\varepsilon(x) = f(x, x/\varepsilon) \quad \text{on } \partialOmega,
\]
where $f(x, y)$ is 1-periodic in $y$, and Neumann problem
\[
\mathcal{L}_\varepsilon(v_\varepsilon) = 0 \quad \text{in } \Omega, \quad \text{and} \quad \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} = T_{ij} \cdot \nabla\{g_{ij}(x, x/\varepsilon)\} \quad \text{on } \partialOmega,
\]
where $T_{ij} = n_i e_j - n_j e_i$ is a tangential vector field on $\partialOmega$ and $\{g_{ij}(x, y)\}$ are 1-periodic
in $y$.

Under the assumptions that $A$ is smooth and 1-periodic, and $\Omega$ is a smooth and strictly
convex domain in $\mathbb{R}^d$, it was proved in [2] that the homogenized problem for (1) is given by
\[
\mathcal{L}_0(u_0) = 0 \quad \text{in } \Omega, \quad \text{and} \quad u_0 = \overline{f} \quad \text{on } \partialOmega,
\]
where $\mathcal{L}_0$ is the usual homogenized operator and $\overline{f}$ is a function whose value at
$x \in \partialOmega$ depends only on $A$, $f(x, \cdot)$ and the outward normal $n$ to $\partialOmega$ at $x$. Similarly,
it was proved in [3] that if $\Omega$ is smooth and strictly convex, the homogenized problem
for (2) is given by
\[
\mathcal{L}_0(v_0) = 0 \quad \text{in } \Omega, \quad \text{and} \quad \frac{\partial v_0}{\partial \nu_0} = T_{ij} \cdot \nabla\overline{g}_{ij} \quad \text{on } \partialOmega,
\]
where $\frac{\partial v_0}{\partial \nu_0}$ denotes the conormal derivative of $v_0$ associated with $\mathcal{L}_0$, and $\{\overline{g}_{ij}\}$ are
functions on $\partialOmega$ whose values at $x \in \partialOmega$ depend only on $A$, $\{g_{ij}(x, \cdot)\}$ and $n(x)$.

Then, it was proved in [1] that

**Theorem 1** [Dirichlet Data] Assume that $A$ is elliptic, smooth and 1-periodic. Let $\Omega$ be a smooth and strictly convex domain in $\mathbb{R}^d$. Let $\overline{f}$ denote the homogenized data in (3). Then
\[
\|\overline{f}\|_{W^{1,p}(\partialOmega)} \leq C_p \left(\int_{\mathbb{T}^d} \|f(\cdot, y)\|_{C^k(\partialT^d)}^p dy\right)^{1/2} \quad \text{for any } 1 < p < \infty,
\]
where $C_p$ depends only on $d$, $m$, $\lambda$, $p$, and $\|A\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d, p) > 1$. 

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Type: Partial Solution

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Theorem 2 [Neumann Data] Assume that $A$ is elliptic, smooth and 1-periodic. Let $\Omega$ be a smooth and strictly convex domain in $\mathbb{R}^d$. Let $\mathcal{g} = (g_{ij})$ denote the homogenized data in (4). Then

$$\|\mathcal{g}\|_{W^{1,p}(\partial\Omega)} \leq C_p \left( \int_{\mathbb{T}^d} \|g(\cdot, y)\|^2_{C^1(\partial\Omega)} \, dy \right)^{1/2}$$

for any $1 < p < \infty$, where $C_p$ depends only on $d$, $m$, $\lambda$, $p$, and $\|A\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d,p) > 1$.

The proofs for Dirichlet and Neumann are similar. The ingredients come from three parts: 1. Maximal principle for solutions in half-spaces; 2. Weighted estimates in half-spaces; 3. An interpolation argument that combines all these estimates. We also point that the results in Theorem 1 and 2 may be extended to domains of finite type considered in [4].

References

https://arxiv.org/abs/1707.03160


https://arxiv.org/abs/1610.05273

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The Original Problem

Regularity of homogenized boundary condition for divergence type elliptic systems

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Consider the homogenization problem of the elliptic system

$$-\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u(x) = 0, \quad x \in D, \quad (1)$$

in a domain $D \subset \mathbb{R}^d$, $(d \geq 2)$, and with oscillating Dirichlet boundary data

$$u(x) = g\left(x, \frac{x}{\varepsilon}\right), \quad x \in \partial D. \quad (2)$$

Here $\varepsilon > 0$ is a small parameter, and $A = A^{\alpha\beta}(x) \in M_N(\mathbb{R})$, $x \in \mathbb{R}^d$ is a family of functions indexed by $1 \leq \alpha, \beta \leq d$ and with values in the set of matrices $M_N(\mathbb{R})$. For each $\varepsilon > 0$ let $L_\varepsilon$ be the differential operator in question, i.e. the $i$-th component of its action on a vector function $u = (u_1, ..., u_N)$ is defined as

$$(L_\varepsilon u)_i(x) = -\left(\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u\right)_i(x) = -\partial_{x^\alpha} \left[ A^{\alpha\beta}_{ij} \left(\frac{x}{\varepsilon}\right) \partial_{x^\beta} u_j\right],$$

where $1 \leq i \leq N$.

Consider (1) under the following conditions:

**(Ellipticity)** there exists a constant $\lambda > 0$ such that $\forall x \in \mathbb{R}^d$, $\forall \xi = (\xi_i^\alpha) \in \mathbb{R}^{dN}$ one has

$$\lambda |\xi_i^\alpha|^2 \leq A^{\alpha\beta}_{ij}(x) \xi_i^\alpha \xi_j^\beta \leq \frac{1}{\lambda} |\xi_i^\alpha|^2.$$  

**(Periodicity)** $A$ and, $g$ in its second variable, are both $\mathbb{Z}^d$-periodic, i.e. $A(y + h) = A(y)$, and $g(x, y + h) = g(y)$ for all $x \in \partial D$, $y \in \mathbb{R}^d$, and $h \in \mathbb{Z}^d$.

**(Smoothness)** The elements of $A$, the function $g$ in both variables, and the boundary of $D$ are $C^\infty$ smooth.

**(Geometry)** Domain $D$ is strictly convex.

For each $\varepsilon > 0$ let $u_\varepsilon$ be the unique (smooth) solution to (1). The main result of [1] states that under the conditions listed above, there exists an $L^\infty$ function $g_* : \partial D \to \mathbb{R}^N$, such that if $u_0$ is the solution to the Dirichlet problem with operator tensor $A^0$ (the classical homogenized coefficients), and boundary data $g_*$, then for any $0 < \alpha < \frac{d-1}{3d+5}$ one has

$$||u_\varepsilon - u_0||_{L^2(D)} \leq C_\alpha \varepsilon^\alpha,$$

where the constant $C_\alpha = C(\alpha, D, A, g, d)$. This breakthrough result in the analysis of homogenization of (1)-(2) gives rise to the following natural question:

**What is the regularity of the homogenized boundary condition $g_*$?**
The function $g_*$ in [1] is defined at all $x \in \partial D$ with Diophantine normal vector, where a unit vector $n \in \mathbb{R}^d$ is called Diophantine if there exist constants $\kappa, l > 0$ such that $||P_{n^\perp}(\xi)|| \geq \kappa ||\xi||^{-l}$ for all non-zero $\xi \in \mathbb{Z}^d$, where $P_{n^\perp}$ is the projection operator on the direction orthogonal to $n$. It is not hard to see that for any fixed $l > 0$ satisfying $l(d - 1) > 1$ almost all points (with respect to the $\mathcal{H}^{d-1}$-measure on the sphere) are Diophantine with some constant $\kappa > 0$ (the constant $\kappa > 0$, however, is not bounded away from 0). Thus, $g_*$ is defined almost everywhere on the boundary of $D$.

To outline how Diophantine condition comes into play, we next bring up the notion of boundary layer systems introduced in [2].

For a unit vector $n$, consider the following system

\[
\begin{align*}
-\nabla_y \cdot A(y)\nabla_y v(y) &= 0, & y \cdot n > 0, \\
v(y) &= v_0(y), & y \cdot n = 0
\end{align*}
\]

where $v_0$ is smooth and $\mathbb{Z}^d$-periodic (and when applied to (1)-(2) is defined via $g$ - the original boundary data). Systems of the form (3) were introduced and studied in [2], and later in [1], and play a central role in the analysis of (1)-(2). It was proved in [1] (see also [2]) that under the Diophantine condition on the normal $n$, the solution to (3) converges as $y \cdot n \to \infty$ to a constant vector field named as a boundary layer tail. The homogenized boundary condition $g_*$ is defined via the function $x \mapsto v_\infty(n(x))$ where $x \in \partial D$ and has a Diophantine normal vector, and $v_\infty$ is the boundary layer tail corresponding to $n$. Hence the regularity of $g_*$ is boiled down to understanding the regularity of boundary layer tails with respect to the normal vector field of $\partial D$.

It is proved in [1] that boundary layer tails are Lipschitz continuous, however, the Lipschitz constant blows up (as the Diophantine properties of the normal vectors deteriorate). From the (non-uniform) Lipschitz estimate it follows that $g_*$ is continuous at all points of $\partial D$ with Diophantine normal vector. But since the Lipschitz bounds on boundary layer tails are not uniform along $\partial D$, it is not clear, for example, if $g_*$ admits continuous extension to all points of $\partial D$ (recall that $g_*$ was defined only at points with Diophantine normals).

Understanding the regularity of $g_*$ presents a challenging mathematical question on its own right, and may lead to a better understanding of homogenization of (1)-(2).

**References**
